On the Structure of the Pantoum

1 Form

There is only one essential feature of the pantoum: repeating lines. The rest is variation. The standard pattern, observed throughout the preceding collection, repeats the second & fourth lines of a given stanza as the first & third of the next, with the third & first lines of the initial stanza reappearing as the second & fourth of the final. Thus in each pantoum of this collection, whether of three stanzas or four, every line was used exactly twice. All these pantoums belong to a general class of poems which we will denote the $(N, 2N, 4N)$ pantoums because for each of their N stanzas they have $2N$ unique lines & $4N$ total lines.[1] We will see that the $(N, 2N, 4N)$ pantoum is only one among an infinite number of classes of pantoums.

1.1 The Generalized Pantoum

The general pantoum may consist of any number of lines. As stated above, it is only essential that lines repeat. All lines may be written twice, as in the $(N, 2N, 4N)$ pantoum, or other rules of greater or lesser complexity may be introduced: the pantoum may consist of an indefinite number of lines

[1] Some parts of this triple specification may seem redundant, but it serves to describe several features of the poem concisely. Taken together, the second & third parts of the triple specify how many times each unique line must repeat ($4N/2N = 2$), while the first & third parts ensure that stanzas are regular (having, in this case $4N/N = 4$ lines each). With this notation we may also specify nonstandard forms such as the $(N, 3N, 12N)$ pantoum, in which $3N$ unique lines repeat 4 times each over the course of N regular 12-line stanzas.

repeated according to chance, or of a single line repeating indefinitely. Yet however simple or arcane, the rules which call for the repetition of a line amount in the end to a function. This function is defined over the set T of total lines, & gives, for any line $t \in T$, the unique line u which is to be written down. For example, one may specify a particular class of pantoums by

$$P(t) = \sin\left(\frac{t\pi}{2}\right) + t^2 \bmod |U|, \quad 1 \leq t \leq |T|$$

where $|U|$ is the size of the set U of unique lines. Then $P : T \to U$ specifies which unique line $u \in U$ will appear on any line t. If we let $T = \{1, 2, \ldots, 10\}$ & $U = \{1, 2, \ldots, 5\}$, P yields a pantoum of the following form:

3 THE PARK IS A FLAT GREEN
5 WHERE THE SUN BREAKS IN
4 TO CRAWL FROM THE TREES
2 AND THE SHADOWS LONG TOO EARLY
2 AND THE SHADOWS LONG TOO EARLY
2 AND THE SHADOWS LONG TOO EARLY
4 TO CRAWL FROM THE TREES
5 WHERE THE SUN BREAKS IN
3 THE PARK IS A FLAT GREEN
1 THAT MUTES THE SHAPELY NIGHT

I will not insist that this pantoum is a good one (its lines relate an impression gathered from a walk I took searching for things to stuff into this form), but it highlights a number of important things about the pantoum generally.

Firstly, the only requirement for the function $P : T \to U$ which determines a pantoum is that it be surjective[2]; all its other features are left to the judgment of the poet. Since T & U may be any sets of natural numbers where $|T| > |U|$, there are an infinite number of functions satisfying this

[2] Each unique line $u \in U$ must correspond to at least one line $t \in T$. Since lines repeat, $|T| > |U|$ & there is at least one u for which $P(t_1) = u = P(t_2)$, where $t_1, \neq t_2$.

condition. Secondly, the lines of a pantoum need not be numbered in order of appearance. In the example above, we begin on line "3", & do not reach line "1" until the end of the poem. The numbering of the lines, like the function determining their position, is arbitrary. Finally, a word on composition. The lines of the pantoum must repeat, but it is best they not repeat too often. Patterns re-worked over a short span of lines may weary the reader (as perhaps does the triple-repetition of line 2 in the pantoum above), & if it begins to seem that every other line is the same, anyone who values their time will skip over all that wasted ink. For similar reasons, such lines as do repeat are best kept reasonably short.

1.2 Stanzaic Shift

The $(N, 2N, 4N)$ pantoum adds one additional restriction to the function P. Since stanzas are regular & lines from stanza n are repeated in stanza $n + 1$, stanza length must be accounted for in P's construction. For convenience, let us define the stanza function

$$S(t, s) = \lceil \frac{t}{s} \rceil$$

where t is any of the T total lines, s is the number of lines in each stanza, & $\lceil \frac{t}{s} \rceil$ is the least integer z for which $z \geq \frac{t}{s}$. For instance, when $s = 4$, the eleventh line of the poem falls in stanza $S(11, 4) = \lceil \frac{11}{4} \rceil = 3$. Including S as a component of our pantoum function P is one way of fixing the stanza of a given line t. What is left is to specify how the unique lines shift as they move from stanza to stanza. This can be handled by an additional component R, so that our function P for a pantoum of regular s-line stanzas takes on the form

$$P(t, s) = aS(t, s) + bR(t, s)$$

where a & b are constants.

1.3 Functional Description of the (N, 2N, 4N) Pantoum

Let us conclude this section with a function P for the $(N, 2N, 4N)$ pantoum, which we here define more formally. In the following, a line is said to be new if it has not appeared previously in the text of the poem; it is said to be repeated if it is not new.

Definition 1.1. *By the symbols $(N, 2N, 4N)$ we denote the class of pantoums which are composed according to the following rules:*

1. ***(Rule of Stanzas)*** *The pantoum has N stanzas, where N is an integer greater than 1.*
2. ***(Rule of Regularity)*** *All stanzas have 4 lines.*
3. ***(Rule of New Lines)*** *For all stanzas $n < N$, the second & fourth lines of n are new.*
4. ***(Rule of Repeated Lines)*** *For all stanzas $n < N$, the second & fourth lines of n are repeated as the first & third lines of $n + 1$, respectively.*
5. ***(Rule of Initial Stanza)*** *The first & third lines of stanza 1 are new.*
6. ***(Rule of Final Stanza)*** *The third & first lines of stanza 1 are repeated as the second & fourth lines of stanza N, respectively.*

We will make use of the 4-line stanza function $S(t, 4)$, so that P is of the form

$$P_{N,2N,4N}(t, 4) = aS(t, 4) + bR(t, 4)$$

where a & b are constants as above. To address the Rule of Repeated Lines (4), we define a cyclical function R as

$$R(t, s) = (-1)^t + \frac{1}{2}\left(\left((t-1)^2 - (t-1)\right) \bmod s\right), \quad 2 \leq t \leq |T|.$$

For $s = 4$, R yields the repeating sequence of values $\{-1, 1, 0, 2\}$. With this function to determine the relative position of lines within each stanza, we need only specify further that the pantoum start with unique line 1, & that 2 lines from the first stanza reappear in stanza N. We write the full pantoum function $P_{N,2N,4N}$ as

$$P_{N,2N,4N}(t,4) = \begin{cases} 1 & \text{if } t = 1 \text{ or } t = 4N, \\ 2 & \text{if } t = 4N - 2, \\ 2S(t,4) + R(t,4) & \text{otherwise.} \end{cases}$$

Using this function & a slightly altered method of numbering lines, we can generate an example of the $(N, 2N, 4N)$ pantoum from the unique lines of Pantoum №7 in the preceding collection:

1 IN THE BODY OF THE AFTERNOON
3 THE TREEBARK LINES MY BACK THE GRASS MY LEGS
2 THE WEAVING OF THE NET IS PRESSED IN FLESH SO
4 GRAVES CRY OUT THEIR NAMES THROUGH CARVINGS SCARS

3 THE TREEBARK LINES MY BACK THE GRASS MY LEGS
5 THE DAY GROWS ON BUT I SINK INTO DREAMS WHERE
4 GRAVES CRY OUT THEIR NAMES THROUGH CARVINGS SCARS
6 AND VIRGINS BURN LIKE TORCHES FOR THE NEARING

5 THE DAY GROWS ON BUT I SINK INTO DREAMS WHERE
7 GOD DRAWS IN THE DARK THAT HOLDS THE STARS
6 AND VIRGINS BURN LIKE TORCHES FOR THE NEARING
8 FALL THE MISSILES OF THE CHOKING NIGHT

7 GOD DRAWS IN THE DARK THAT HOLDS THE STARS
2 THE WEAVING OF THE NET IS PRESSED IN FLESH SO
8 FALL THE MISSILES OF THE CHOKING NIGHT
1 IN THE BODY OF THE AFTERNOON

As stated above, the unique lines need not be numbered in order of appearance. Swapping 2 & 3 allows us to specify all the lines of stanzas

$\{2, 3, \ldots, N-1\}$ with the single expression[3] $2S + R$. Given this method of numbering & the $2N = 8$ unique lines of our example, $P_{N,2N,4N}$ has arranged for us a pantoum adhering to rules 1 - 6. Incidentally, this form also agrees with our rules of composition: lines repeat only once with a sufficient number of lines between repetitions.

2 Representation

Having examined analytically some properties of the pantoum, let us now turn to its graph. This section of our investigation will be limited to the $(N, 2N, 4N)$ pantoum, although it is easily adapted to nonstandard forms.

2.1 The Neighbors of a Pantoum Line

A function may dictate where & how often the lines of a pantoum will appear, but of greater interest to the reader is the varying (or unvarying) context of each line. A set of lines repeating in the same order is not particularly compelling, but when old lines begin to intrude on new neighbors, their potential to carry several meanings is realized. The rules of the $(N, 2N, 4N)$ pantoum are well-suited to ensure this contextual shuffling. This is easily seen when the poem is represented as a graph. Let us define the graph of a pantoum as follows:

Definition 2.1. *A pantoum graph is given by the pair $G(U, E)$, where the set of nodes U consists of all unique lines in the poem & the set of edges E consists of all distinct pairs of lines which are adjacent in the text (ignoring stanza boundaries).*

When we are concerned about the order of lines in the text, we may further state

[3] It is likely possible to specify the full pantoum by a single closed-form expression which accounts for the reversed repetition of unique lines 1 & 2, & the fact that $R(1, 4)$ is not defined on T. But I am neither clever nor determined enough to formulate this expression, & in the end it is equivalent to the given conditional form. Rather than hide the exceptions of the first & last stanzas with dubious computational tricks (powers of 0 come to mind, or the inclusion of lines in T which never appear in the poem), I have elected to make them explicit.

Definition 2.2. *A directed pantoum graph is given by $G\left(U, \overline{E}\right)$, where the set of nodes U is as stated in Definition 2.1 & the set of directed edges $\overline{E}$ consists of all distinct ordered pairs of lines adjacent in the text (ignoring stanza boundaries).*

A directed pantoum graph is shown in Figure 1.1 for the $(N, 2N, 4N)$ pantoum of $N = 4$ stanzas. Each node represents one of the 8 unique lines in the poem, while a directed edge between nodes u_1 & u_2 indicates that u_1 immediately precedes u_2 in the text.

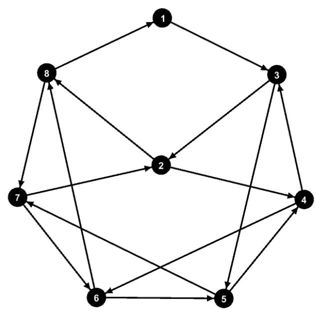

Figure 1.1: The graph of the $(4, 8, 16)$ pantoum.

As this graph shows, each unique line of the $(4, 8, 16)$ pantoum is adjacent to fully half of the other unique lines over the course of the poem's development, with the notable exception of line 1. This suggests that lines which do not form complete "units of thought" must have considerable flexibility, as they will combine syntactically & semantically with new lines when repeated. In fact, when the number of stanzas $N > 2$, the graph of the $(N, 2N, 4N)$ pantoum cannot be drawn without intersecting edges (i.e., the graph is nonplanar, which is proved in the appendix).

If we view these necessary intersections as a confusion or mixing of contexts—& here, perhaps, we stretch the analogy thin—we see that the form of the $(N, 2N, 4N)$ pantoum creates a rolling sea of meanings on which the poet must float his central conceit. The repeating structures of the middle stanzas are waves which push us out from shore & draw us in again, while the launch & landing of our voyage is the selfsame harbor, the unique line 1. There is something Heraclitean in these waters, for although the words repeat, their meanings are never the same. Our path is always forward, yet many times self-crossing, & when we return to harbor it is not by the way we came.

3 Closing Remarks

There is only one essential feature of the pantoum: repeating lines. The rest is variation. We have seen that any number of variations on the form may be generated by suitable functions. The function we gave to describe the familiar pantoum involved some computational complexities stemming from stanzaic shift & the reversed reappearance of lines in the final stanza, yet we saw that it specifies a poetic form which adheres to the principles of composition advanced in this essay. Further, this form has an internal structure which encourages a blending & transformation of meanings, a revelation of the possible interpretations of each unique line. Of course, the poet may avoid this contextual recombination by clever punctuation, or by employing only the simplest constructions of grammar. But this is to retreat from the potential of the form. A strong writer of pantoums will rather embrace this complexity & seek to transform the first-perceived meaning of a line through its repetition. In fact, I take this to be a goal of pantoum composition: the meaning of a given line should develop as it is repeated across stanzas. In the $(N, 2N, 4N)$ pantoum, the poet should seek to achieve this locally (through lines like 3 & 4 which appear in adjacent stanzas) as well as globally (through the last-stanza repetition of lines 2 & 1). A pantoum which falls short of this mark has static or at best repeating structures of meaning, but one which hits the target grows its significance continually over the lines of the poem.

This investigation may seem tedious to the poet uninterested in number. Yet the application of mathematics to verse forms opens a window onto the structure of those forms, so that the poet whose eye is trained to spot potential in such structures may use mathematics to the profit of his craft.

There is also a poetry, in the etymological sense of creating or making, in the very search for mathema-poetical structures. Their bodies are formed from the dust of numbers, spirit breathed in by word. I give no estimate of how many interesting—much less enduring—forms may be discovered with the techniques suggested here. The examples given, & even the methods themselves are far from definitive. Some poet-mathematician of greater wit & wisdom than myself will come along to enlarge for us all the common wealth of verse; I leave here a few open questions with which such a one might begin.

1. The repeating lines of the pantoum & their recombination across stanzas remind one of cyclic groups. Yet a group operation is difficult to define for the $(N, 2N, 4N)$ pantoum. And further, any such operation would be applied to only a fraction of the possible line pairs over the course of the poem. Which pantoum forms are better adapted to the structures of group theory or abstract algebra?

2. We have stated that the $(N, 2N, 4N)$ pantoum is nonplanar for $N > 2$, & that this may have implications for how the lines of the poem are written. What additional features of pantoum graphs create interesting challenges for the poet?

3. Since the graph of the $(N, 2N, 4N)$ pantoum is nonplanar for $N > 2$, it cannot be embedded in a surface of genus 0. On what surface(s) can this graph be drawn?

Appendix

A

Below we prove Theorem A.4, which states that whenever the number of stanzas N is greater than 2, the graph of the $(N, 2N, 4N)$ pantoum cannot be drawn without intersecting edges. Let us first establish a preliminary result about the number of unique lines in the $(N, 2N, 4N)$ pantoum.

Lemma A.1. *The $(N, 2N, 4N)$ pantoum has $2N$ unique lines.*

Proof. To count the number of unique lines, we begin in stanza 1 & count all lines which are new. By the Rule of Initial Stanza (5) in Definition 1.1, the first & third lines of stanza 1 are new. By the Rule of New Lines (3), the second & fourth lines of all stanzas $n < N$ are also new. Since there are $N - 1$ such stanzas, there are at least $2 + 2\,(N - 1) = 2N$ new lines in the first $N - 1$ stanzas of the $(N, 2N, 4N)$ pantoum.

By the Rule of Stanzas (1) & the Rule of Regularity (2), there are $4\,(N - 1)$ total lines in the first $N - 1$ stanzas. But by the Rule of Repeated Lines (4), there are 2 repeated lines in each of the stanzas $\{2, 3, \ldots, N - 1\}$, so that there are at most $4\,(N - 1) - 2\,(N - 2) = 4N - 4 - 2N - 4 = 2N$ new lines in the first $N - 1$ stanzas (there are no repeated lines to exclude from stanza 1, since by the Rule of New Lines (3) & the Rule of Initial Stanza (5) all 4 lines are unique). Thus, denoting the number of new lines in the first $N + 1$ stanzas by u, we have $u \geq 2N$ & $u \leq 2N$, from which it follows that

$u = 2N$. It remains only to check stanza N for additional new lines. But from the Rule of Repeated Lines (4) & the Rule of Final Stanza (6), it follows that stanza N contains 4 repeated lines & therefore no new ones. Hence the $(N, 2N, 4N)$ pantoum contains exactly $2N$ unique lines, q.e.d.

From this lemma & the definition of a pantoum graph given in Section 2.1 of the preceding essay, it follows directly that

Lemma A.2. *The graph of the $(N, 2N, 4N)$ pantoum, denoted $G_{N,2N,4N}$, has $2N$ nodes.*

As a last preparatory step, we will use Lemma A.2 to demonstrate one further result.

Lemma A.3. *Let $G_{N,2N,4N}$ denote the graph of the $(N, 2N, 4N)$ pantoum, & let $G_{N+1,2(N+1),4(N+1)}$ denote the graph of the $(N+1, 2(N+1), 4(N+1))$ pantoum. Then $G_{N+1,2(N+1),4(N+1)}$ contains $G_{N,2N,4N}$ as a minor.*

Proof. For some $N > 1$, we begin with an $(N+1, 2(N+1), 4(N+1))$ pantoum p composed according to Definition 1.1. We then label the stanzas of p sequentially from 1 to $N+1$. Choosing any stanza n such that $1 < n < N+1$, we label the lines of n with subscripts n_1 through n_4 in order of appearance. We now make the following alterations to the text of p:

1. Replace the second & fourth lines of stanza $n-1$ with lines n_2 & n_4, respectively.

2. Remove stanza n from the poem.[1]

We will show that the resulting pantoum p^* adheres to rules 1 - 6 of Definition 1.1.

Clearly p^* respects the Rule of Stanzas (1) & the Rule of Regularity (2), since it contains N stanzas of 4 lines each. To verify that p^* follows the Rule of New Lines (3), we need only check stanza $n-1$, as stanza n was

[1]When removing stanza n, we do not relabel the stanzas, so that for all stanza labels l_p assigned to the stanzas of p, $l_p \neq n$ implies l_p is also assigned to a stanza of p^*.

removed & no additional stanzas were modified. Since n_2 & n_4 were new in stanza n of the pantoum p, they must also be new in stanza $n-1$ of p^*; otherwise, they would have appeared in one of the stanzas $\{1,\ldots,n-1\}$ of p, contrary to Definition 1.1. Therefore p^* adheres to the Rule of New Lines (3). Similarly, p^* respects the Rule of Repeated Lines (4). In p, lines n_2 & n_4 were repeated as the first & third lines of stanza $n+1$; they still occupy these positions in p^*, & since stanza n was removed, n_2 & n_4 are repeated in the stanza immediately following their first appearance, as required. Finally, p^* adheres to the Rule of Initial Stanza (5) & the Rule of Final Stanza (6) because neither the first or third line of stanza 1, nor the second or fourth line of stanza $N+1$, were modified by our changes to the text. Thus p^* is a valid $(N, 2N, 4N)$ pantoum.

Since p^* is an $(N, 2N, 4N)$ pantoum, it follows by Lemma A.2 that the graph of p^*, an instance of $G_{N,2N,4N}$, has $2N$ nodes. Further, between the $4N$ unique lines of p^*, there are a total of $4N-1$ pairs of adjacent lines.[2] In other words, $G_{N,2N,4N}$ has $4N-1$ edges. Compared to the graph of p, an instance of $G_{N+1,2(N+1),4(N+1)}$, we see that $G_{N,2N,4N}$ has $2(N+1)-2N = 2$ fewer nodes & $(4(N+1)-1)-(4N-1) = 4$ fewer edges. Hence $G_{N,2N,4N}$ can only be formed from $G_{N+1,2(N+1),4(N+1)}$ by deleting or contracting edges, or by deleting nodes. $G_{N,2N,4N}$ is therefore a minor of $G_{N+1,2(N+1),4(N+1)}$, q.e.d.

With the help of the above lemmas, we conclude with a proof of

Theorem A.4. *The graph of the $(N, 2N, 4N)$ pantoum is nonplanar for all $N > 2$.*

Proof. We begin with $N = 3$ & the graph $G_{3,6,12}$, as shown in Figure 2.1. This graph has 6 nodes. Note that node 1 has degree 2, nodes 3 & 6 have degree 3, & the remaining nodes have degree 4. If we contract node 1 into node 3 to form a minor of the graph, the edge which formerly joined node 1 & node 6 will join node 6 & node 3, so that these latter nodes are added to the subset of nodes having degree 4. There are then 5 such nodes in the

[2]This can be observed by counting the number of spaces between lines in p^*, since adjacent lines cannot be separated by any text.

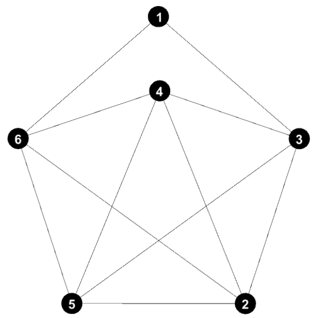

Figure 2.1: The graph of the $(3, 6, 12)$ pantoum.

minor, as shown in Figure 2.2. In other words, $G_{3,6,12}$ has a minor of $k = 5$ nodes, each of degree $k - 1 = 4$. Since this minor is isomorphic to K_5, we have by Wagner's theorem that $G_{3,6,12}$ is nonplanar.

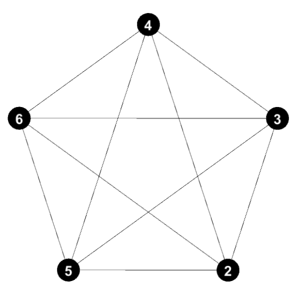

Figure 2.2: $G_{3,6,12}$ has a minor (isomorphic to) K_5.

Next, assume $G_{N,2N,4N}$ is nonplanar. By Lemma A.3, $G_{N+1,2(N+1),4(N+1)}$ contains $G_{N,2N,4N}$ as a minor. Since $G_{N,2N,4N}$ is nonplanar, by Wagner's theorem this minor must itself contain a minor isomorphic to K_5 or $K_{3,3}$. Therefore $G_{N+1,2(N+1),4(N+1)}$ also contains a minor isomorphic to K_5 or

$K_{3,3}$, & is consequently nonplanar.

We have shown that $G_{N,2N,4N}$ is nonplanar when $N = 3$, & that if $G_{N,2N,4N}$ is nonplanar, then $G_{N+1,2(N+1),4(N+1)}$ is also nonplanar. By mathematical induction, we conclude that $G_{N,2N,4N}$ is nonplanar for all $N > 2$, q.e.d.